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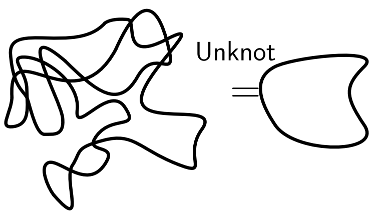
Untie the Knot!

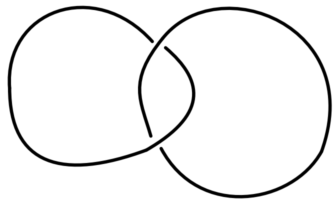
by Prognadipto Majumder

"In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics." - Hermann Weyl

Introduction

We instinctively know what a *knot* is. This article will be a tale of how knots are viewed through the lens of topology, and how a generation of mathematicians has worked to study them from a purely algebraic standpoint. For our purposes, we can agree upon a knot to be a string, possibly twisted or knotted, with no loose ends. A *link* is defined to be some number of knots arranged in some specific way in 3D space.





Hopf Link

In Greek mythology, an oracle claimed that whoever was able to untie an intricate knot tied by King Gordius would become the next ruler of Asia. Alexander the Great, unable to untie the knot, became impatient and swiftly cut it with his sword.

Initially, the best we can do is twist and turn with the knots without breaking or collapsing any of the strings. This ultimately is a topological question in mathematics. In the words of Edward Witten,

> "In Einstein's general relativity the structure of space can change but not its topology. Topology is the property of something that doesn't change when you bend it or stretch it as long as you don't break anything."

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Later, we would find ways to cut up knots and re-glue them to study knots in an algebraic fashion. Mathematicians are simple folks, with a very generic philosophy towards life: give a man a fish, he will eat for a day; give a mathematician a fish, and he will try to define axiomatically what a fish is; next, he will aim to classify all fish and prove theorems regarding when to call two fish of the same kind.

The first serious study of knots arose when Lord Kelvin hypothesized that different molecules are different kinds of knots formed by stringlike vortices (of what?) in aether. In a rush to classify all molecules, Peter Guthrie-Tait jumped to describe all knots up to a given complexity, so that the first serious mathematical discussion on knot theory was done by a chemist.

Fig: Peter Guthrie Tait's tabulation of knots and their properties (has lots of redundancies!).
Peter Guthrie Tait

Classification of knots

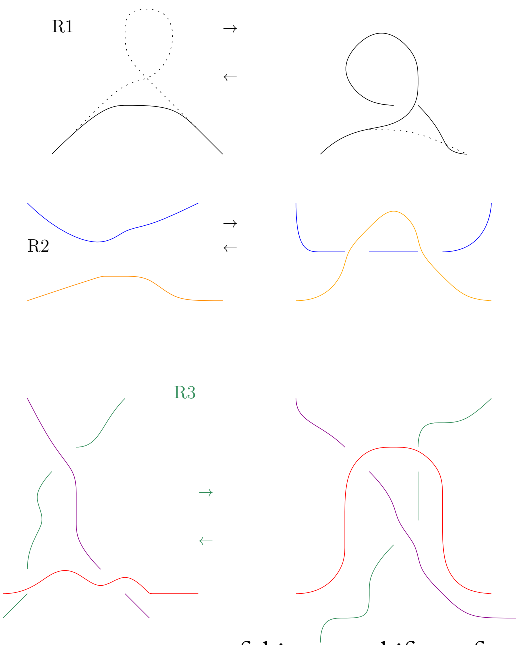
At this point, we run into one of the basic underlying philosophies of mathematics: how do we represent a knot mathematically, how do we study them, and when do we call two knots the same?

Knots are objects in 3D space, so when we look at them from one side and project them on a flat piece of paper, we need to take note of which string lies above and which one lies below. For simplicity, we assume that no three segments of the string 'intersect' at the same point when viewed as a projection onto some plane. Given such a diagram, we can construct the full knot in 3D space, so we have not lost any information. The difficulty begins when we look at a knot from two different ways- they might look like two different projections. We call two projections to be 'the same' if we can transform one into the other by some smoothly varying moves that keep the crossings invariant. One of the first important steps was taken by Kurt Reidemeister (and independently by James Waddell Alexander and Garland Baird Briggs), who proved that any two projection diagrams represent the same knot if and only if they can be related by a sequence of three very special moves called Reidemeister moves:

R1: Twist and untwist in either direction.

R2: Move one loop completely over another.

R3: Move a string completely over or under a crossing.



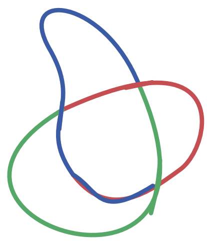
Now, proving two things are the same is simple in theory; we just need to algorithmically find out a sequence of steps between two such knot diagrams. Brute force computation becomes useless due to the sheer size of computations, in which regard it might be interesting to note a result by Coward and Lackenby[1].

Theorem 1 *Let D and D' be link diagrams and suppose that D' is obtained from D by the addition or removal of an unknot summand. Suppose that D and D' both have at most n crossings. Then there is a sequence of at most $2^{10^{11}n}$ Reidemeister moves relating D and D'.*

So, we ask a simpler question, how to show when two knots are NOT equivalent? To get a concrete grasp of things, we shift our focus to two questions:

- Given a trefoil knot, is it possible to untie it to get a ring?
- Given a Hopf link, is it possible to separate the two rings without tearing them apart?

These two questions would give us an idea of *knot invariants*, which will be the main theme of our story hereafter, and which we will conclude with a mention of Vaughan Jones and Mikhail Khovanov.



Tricoloured Trefoil

To solve the second question, Gauss thought of the idea of what the induced magnetic field would be in the second loop when a current flows through the first. This is 0 for two non-linked rings and nonzero for the Hopf link.

For the first question, we define a tricolouring of a knot to be a coloring with three colors such that each strand receives one color, at least

[1]Coward, Alexander, and Marc Lackenby. "*An upper bound on Reidemeister moves.*" American Journal of Mathematics 136.4 (2014)

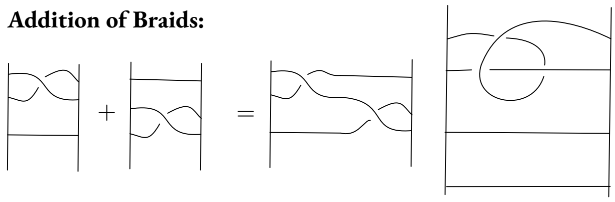
two colors are used in total, and at each crossing, the strands are either all the same color or all different colors. Using this we can show tricolourability is a knot invariant; and whereas the trefoil knot is tricolorable, the unknot (which we call the ring with no crossings or twists) is NOT.

Classical knot theory gave rise to a plethora of knot invariants, each to address a very specific question or related to some very restricted class of knots. Over the course of years, a large number of simple intuitive questions regarding classification of knots began to be posed, without any strong technique generic enough to address all or most of these questions in one solid framework.

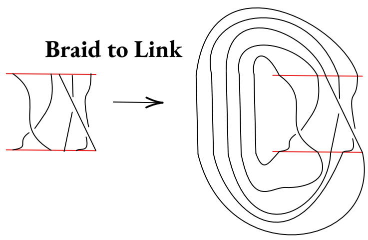
Jones Polynomial

Probably the first gigantic breakthrough came through the work of Vaughan Jones, for which he was later awarded the Fields medal (1990) which he accepted wearing the New Zealand rugby jersey. A functional analyst from New Zealand, Jones came across some nice representations of Artin braid groups while studying type II_1 algebra towers in his foundational paper 'Index for subfactors'[2].

We will look at braids, which we can think of as strings with fixed endpoints at the top and bottom of the figure, each segment goes from some top peg to some bottom peg, and might be twisted in any way. The braids with a fixed number of pegs admit a kind of non-commutative 'addition', namely concatenation, and the group of all braids on n+1 is a well-known object in group theory, the braid group B_n. James Alexander showed that every link can be obtained by closing the ends of some braid. The works of Markov showed that two links are equivalent if and only if their underlying braids are related by a sequence of braid moves, well-known in group theory. Jones found that using the trace function of the representation, there was a polynomial associated with every braid.



This is not a braid.
All strings should go from left to right.



[2]Available at https://link.springer.com/article/10.1007/BF01389127.

Kauffman interpreted the construction of the polynomial as a state sum. We can 'resolve' a crossing in one of the two ways so that we don't have intersecting curves. For every crossing, we have two choices, and we get (2^n) states. Jones' polynomial immediately answered many longstanding questions in knot theory, one of them being the question of amphichirality, i.e., when a knot is equivalent to its mirror image. Yet several questions remained unanswered. One such question was, is it true that the Jones polynomial being 0 implies the knot is topologically same as unknot? While this has a negative answer for links, it is still unknown for knots. Questions like this, and the lack of finer structure in the theory led to the quest of something called categorification in the context of knots, leading to the seminal work by Mikhail Khovanov.

Khovanov Homology

Exposition to Homology: We look at a 1-dimensional chain complex, where we label the groups at each position by a 'degree'. The group at the i-th degree is named C_i.
So we have constructed a much richer structure with maps that helps us to study knots and links far more efficiently. Furthermore, while this complex is not invariant, the successive quotients of image/kernel at each chain degree are indeed invariant, and is known as the Khovanov Homology of the knot or link.
To get an idea of what is homology, we note as an aside, the homology (H^*) of the chain complex associated to a sphere (S^2):

Complex:	$\mathbb{Z} \longrightarrow$	$0 \longrightarrow$	$\mathbb{Z}$
Degrees	2	1	0

Define the kernel to be all elements that map to 0. Then at degree 2, kernel is $\mathbb{Z}$. Then, we define homology at a degree to be the kernel of the outgoing arrow quotiented by the image of the incoming arrow. So, here $H^* \simeq \mathbb{Z}$ if $* = 0, 2$ and 0 otherwise.
Khovanov homology involves a two-dimensional version of the above setup, where we have two different degrees, the chain degree and quantum degree, and the complex looks like a grid.

Categorification is a concept that we won't attempt to define in this exposition, but we can say that it adds more structure to objects and, more importantly, to the maps between objects. For instance, the natural numbers are categorified by n-dimensional vector spaces over a fixed field k, and the increasing relation of numbers translates to a map between vector spaces as the inclusion in the initial coordinates.

Khovanov did something similar with the state-sum model of the knot with n crossings. First, we label the crossings and get 2^n binary strings. Now, for each circle, we assign the vector space V, and each state gets assigned a tensor product of V's. This comes with an inbuilt product and coproduct of V, and we obtain maps between the states that correspond to two circles merging (product in V) or splitting (coproduct in V). This constitutes something called a chain complex, which simply means that any two maps composed turn out to be 0. To add the final touch, we can show that the graded Euler characteristic, an algebraic property of the chain complex, is the Jones polynomial.

So we have constructed a much richer structure, using maps, that helps us study knots and links far more efficiently. Furthermore, while this complex is not invariant, the successive quotients of image/kernel at each chain degree are indeed invariant, and is known as the *Khovanov Homology*[3] of the knot or link.

The study of Khovanov homology has given rise to a lot of beautiful phenomena and results, but to state the very least, we have:

Theorem 2 (*Kronheimer-Mrowka*) *If a knot has trivial Khovanov homology, it is the unknot.*

Dror Bar-Natan explicitly showed that this technique yielded stronger classification power by constructing links with the same Jones polynomial but different Khovanov homologies. The concept of Khovanov homology has opened a deep and rich connection between the topology of knots and the algebraic structures of braid groups. For example, Y. Teng in 2025 used a version of Khovanov homology to find the first known example of an exotic Lagrangian plane in $\mathbb{R}^4$, i.e., a structure in $\mathbb{R}^4$ that is topologically indistinguishable from a 2-d plane but can be distinguished by smooth maps.

Open Questions

There are a couple of questions we can look at (for math enthusiasts):

1. Each of the homology groups (calculated as $\mathbb{Z}$ homology), constitutes of a free part which is a direct sum of copies of $\mathbb{Z}$; and a torsion part, which is some finite abelian group. Now, while the dimension of the free part directly relates to the coefficient of some power of t in the Jones Polynomial, the meaning and consequences of the torsion term is an active area of research. It is a bit of a mysterious object without a well-understood geometric meaning yet. There are a number of conjectures around this, such as:
   - *(Shumakovitch)* The Khovanov homology of every link except the unknot, the Hopf link, their disjoint unions, and their connected sums, has torsion of order 2.
2. Another very useful technique to study knot theory is through knot Floer homology, which was developed independently by Ozsváth-Szabó (Heegaard Floer theory) and Rasmussen around 2002, and which was shown by Ozsváth-Szabó in 2005 to be connected to Khovanov homology via a spectral sequence, a result that finally led to the proof of Kronheimer-Mrowka of how Khovanov homology detects the unknot.

[3]By convention, it should technically be called Khovanov *cohomology*, since chains and their homologies have decreasing chain degrees. When degrees increase along the chain, we call it a cochain and the associated quotients to be cohomology groups.